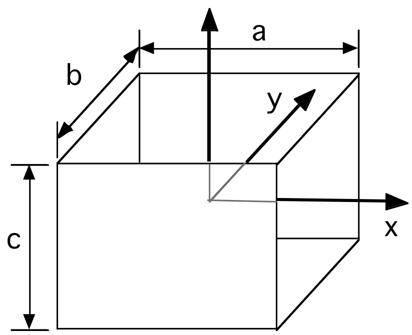
[Thermal-Neutron Spatial Distribution]
(Chapter 1.8.2 in *Elements*)
The spatial distribution of thermal neutrons in a finite medium
The following analysis applies to thermal-neutron moderators (see Chapter 2 in *Elements*). One-speed diffusion theory describes the distribution of well-thermalized neutrons (treated as having a single average speed) nearly isotropically distributed in angle, in nearly isotropic media. The figure illustrates the case of a rectangular parallelepiped-shaped region, the prototypical shape of moderators in pulsed neutron sources.



A parallelepiped-shaped region. The origin of coordinates is at the center of the box.

The diffusion equation that describes the neutron distribution is

$$\frac{1}{v}\frac{\partial \varphi}{\partial t} = D\nabla^2\varphi(\vec{r},t) - \Sigma_{abs}\varphi(\vec{r},t) + S(\vec{r},t), \qquad (1)$$

in which we have suppressed the reference to averages, except to remind that D and Σ_{abs} are spectrum-averaged quantities and that the external source $S(\vec{r},t)$ (independent of φ) must be contrived to represent a distribution with the same spectral shape as $\varphi(v)$. The *Laplacian* operator is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \qquad (2)$$

The equation is a linear partial differential equation in the time and spatial variables, usually treated subject to time-independent extrapolated boundary conditions, which we now discuss without the time dependence,

$$D\nabla^2\varphi(\vec{r}) - \Sigma_{abs}\varphi(\vec{r}) + S(\vec{r}) = 0. \qquad (3)$$

Solving first the *homogeneous equation* (with the external source, $S(\vec{r}) = 0$), which is sometimes referred to as the *spatial wave equation*, or, in nuclear reactor context, as the *reactor equation*,

$$D\nabla^2\varphi(\vec{r}) - \Sigma_{abs}\varphi(\vec{r}) = 0, \qquad (4)$$

which we write

$$\nabla^2\varphi(\vec{r}) - \frac{1}{L^2}\varphi(\vec{r}) = 0, \tag{5}$$

in which

$$L^2 = \frac{D}{\Sigma_{abs}} \tag{6}$$

and L is the *thermal neutron diffusion length*. The boundary conditions on each surface are

$$\bar{\varphi}\left(\vec{r} + \vec{n}d\right)\Big|_{\vec{r}\ on\ surface} = 0, \tag{7}$$

where $\vec{n}$ is the outward-directed normal to the surface and d is the extrapolation distance. The table for [Diffusion Parameters] gives diffusion parameters for several moderating materials at room temperature.

The classic method of solution is the process of *separation of variables*,

$$\varphi(\vec{r}) = \varphi_1(x)\varphi_2(y)\varphi_3(z), \tag{8}$$

in which three independent second-order ordinary differential equations represent the three-dimensional equation, i.e., equations of the type

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Solutions are

$$\varphi_1(x) = A \sin k_1x + B \cos k_1x, \text{ (similarly for y and z),} \tag{12}$$

which clearly satisfy the wave equation for any values of A and B. There are two classes of solutions: even (cosine, symmetric) functions, $A = 0, B \neq 0$, and odd (sine, antisymmetric) $A \neq 0, B = 0$. The boundary conditions (15) establish the values of the k_i's. For the even functions and even integers,

$$\cos\left(\frac{k_1^{\text{even}}\tilde{a}}{2}\right) = 0 \Rightarrow k_1^{\text{even}} = \left(\frac{n_1}{2} + 1\right)\frac{\pi}{\tilde{a}} \text{ for } n_1 = 0,2,4,6, \ldots \tag{13}$$

($k = 0$ is not an eligible eigenvalue because the corresponding function cannot satisfy the boundary condition). For the odd functions and odd integers,

$$\sin\left(\frac{k_1^{\text{odd}}\tilde{a}}{2}\right) = 0 \Rightarrow k_1^{\text{odd}} = \frac{n_1\pi}{2\tilde{a}} \text{ for } n_1 = 1,3,5, \ldots , \tag{14}$$

(using an arbitrary but intuitively attractive numbering scheme). And

$$\tilde{a} = a + 2d \tag{15}$$

is called the *extrapolated width* of the medium. Similarly, $\tilde{b} = b + 2d$ and $\tilde{c} = c + 2d$. The resulting three-dimensional functions are the *spatial eigenfunctions* of the neutron distribution in the medium, which form the basis for further analysis.

The k_i's , which have physical dimensions of inverse length, sum to

$$k_1^2(n_1) + k_2^2(n_2) + k_3^2(n_3) = B_N^2, \tag{16}$$

where B_N^2 is called the *buckling*, or the *eigenvalue*, of the N^{th} eigenfunction,

$$\varphi_N(\vec{r}) = \varphi_{1,n_1}(x)\varphi_{2,n_2}(y)\varphi_{3,n_3}(z) , \tag{17}$$

and $N \equiv (n_1, n_2, n_3)$ represents the triplet of numbers n_1, n_2, and n_3. Moreover,

$$\nabla^2\varphi_N(\vec{r}) = -B_N^2\varphi_N(\vec{r}). \tag{18}$$

The spatial eigenfunctions are a *complete set*. That is, they form the basis for expanding in a series any reasonable function defined in the medium, for example, the flux distribution generated by an external source,

$$D\nabla^2\varphi(\vec{r}) - \Sigma_{abs}\varphi(\vec{r}) + S(\vec{r}) = 0. \tag{19}$$

In this case, the basis functions are sines and cosines, and the series is a *three-dimensional Fourier series*,

$$\varphi(\vec{r}) = \Sigma_N C_N \varphi_N(\vec{r}) , \tag{20}$$

where the C_N are yet-to-be-determined expansion coefficients (constants), so that

$$\Sigma_N C_N\left(DB_N^2 + \Sigma_{abs}\right)\varphi_N(\vec{r}) = S(\vec{r}). \tag{21}$$

Now the crucial property of the eigenfunctions comes into play: the individual functions are *orthogonal*,

$$\int_V \varphi_N(\vec{r})\varphi_M(\vec{r})d^3\vec{r} = A_N\delta_{NM}, \qquad (22)$$

where δ_{NM} is the Kronecker delta and A_N is a *normalizing constant*; that is,

$$A_N = \frac{\tilde{a}\tilde{b}\tilde{c}}{8} = \frac{\tilde{V}}{8}, \qquad (23)$$

where $\tilde{V}$ is the volume within the extrapolated boundaries of the medium.

Multiplying the expanded equation by any arbitrarily chosen eigenfunction $\varphi_M(\vec{r})$ and integrating over the extrapolated volume gives

$$\Sigma_N C_N\left(DB_N^2 + \Sigma_{abs}\right)A_N\delta_{NM} = \int_{\tilde{V}} \varphi_M(\vec{r})S(\vec{r})d^3\vec{r}, \qquad (24)$$

and summing on *N,* with due respect for the Kronecker delta, gives the expansion coefficient for the N^{th} term in the eigenfunction expansion,

$$C_N = \frac{8}{(DB_N^2 + \Sigma_{abs})\tilde{V}} \int_{\tilde{V}} \varphi_N(\vec{r})S(\vec{r})d^3\vec{r}\ , \qquad (25)$$

and completes the calculation of the flux distribution in the medium,

$$\varphi(\vec{r}) = \Sigma_N C_N\varphi_N(\vec{r})\ . \qquad (26)$$