[Single-Nucleus Scattering]
(Chapter 1.3.1 in *Elements*)
Solution of the equation for the scattered wave function
Schrödinger's equation, assuming the unperturbed incident wave function,

$$\nabla_r^2 \Psi_{scattered}(r) - k^2 \Psi_{scattered}(r) \approx \frac{2m}{\hbar^2} \mathrm{V}(r) \Psi_{incident}(\vec{\mathbf{r}}),$$

is a classic inhomogeneous second-order linear partial differential equation whose solution is

$$\Psi_{scatt}(\vec{r}) = \frac{2m}{\hbar^2} \int \frac{\exp\left(ik\left|\vec{r}-\vec{r'}\right|\right)}{4\pi\left|\vec{r}-\vec{r'}\right|} V(\vec{r'}) \exp\left(i\vec{k}\cdot\vec{r'}\right) d^3\vec{r'} = \frac{m}{2\pi\hbar^2} a \frac{\exp(ikr)}{r}, \tag{1}$$

which represents a spherical wave propagating from the origin. The first factor in the solution of the homogeneous equation (the wave function in free space) is the *Green's function* (consult texts on advanced calculus) for the solution of the inhomogeneous equation. From (1) we find

$$b = \frac{m}{2\pi\hbar^2} a, \tag{2}$$

and b is the *bound-atom scattering length*, a constant independent of $\vec{k}$.

The current density (per unit area) in the incident plane wave, $J_{in} = |\vec{J}| = v$, is

$$J_{inc} = \hat{\vec{k}} \cdot \vec{J}_{inc} = hk = v, \tag{3}$$

in which v is the neutron speed, and the total outgoing scattered current (isotropic) is

$$J_{scatt}^{total} = \frac{h}{m} \int_{4\pi} \hat{\vec{r}} \cdot \mathrm{Im}\left[\Psi_{scatt}^* \vec{\nabla} \Psi_{scatt}\right] r^2 d^2\Omega =$$

$$= \frac{h}{m} \int_{4\pi} \mathrm{Im}\left[\Psi_{scatt}^* \frac{d}{dr} \Psi_{scatt}\right] r^2 d^2\Omega = 4\pi \frac{h}{m} k |b|^2. \tag{4}$$

The *microscopic scattering cross-section* is the ratio of the current scattered by a single nucleus to the incident current per unit area,

$$\sigma_{scatt} = \frac{J_{scatt}^{total}}{J_{inc}} = 4\pi \frac{h}{m} k |b|^2 / hk = 4\pi |b|^2, \tag{5}$$

which is independent of $\vec{k}$ and independent of the spatial density of the incoming neutrons. It is the *bound-atom scattering cross section* because its derivation assumes that the struck nucleus is at rest at a fixed position (or, equivalently, has infinite mass) and b is the *bound-atom scattering length*.

Exercise

Verify that the wave functions for plane waves and spherical waves satisfy the Schrödinger equation in free space.