[Lethargy]
(Chapter 1.10.1 in *Elements*)
We rewrite the time-dependent diffusion equation, this time in terms of the *lethargy*, *u*:

$$u \equiv \ln\left(\frac{E_{ref}}{E}\right), \text{ that is, } E = E_{ref}\exp\left(-u\right) \tag{1}$$

and

$$\frac{dE}{du} = -E_{ref}\exp\left(-u\right) \ , \tag{2}$$

in which E_{ref} is an arbitrary reference energy chosen to exceed the energy of any neutrons of interest. Then u increases when the energy decreases and is always greater than zero. The time-dependent diffusion becomes

$$\frac{1}{\mathrm{v}(u)}\frac{\partial\varphi(\vec{\mathbf{r}},u,t)}{\partial t} = D(u)\nabla^2\varphi(\vec{\mathbf{r}},u,t) - \Sigma_{total}\varphi(\vec{\mathbf{r}},u,t) +$$
$$+\frac{1}{(1-\alpha)}\int_{u-\ln\left(\frac{1}{\alpha}\right)}^{u} exp(u'-u)\Sigma_{scat}(u')\varphi(\vec{\mathbf{r}},u',t)\,du' + S(\vec{\mathbf{r}},u,t). \tag{3}$$

In making this transformation, as always in expressing densities in terms of related quantities, it is necessary to include the Jacobian of the transformation, which is in addition to the explicit substitution of the new variable in terms of the old variable. If $f_1(q)$ and $f_2(r)$ are two densities representing the same quantity in terms of two variables, let us call them q and r (now we distinguish the functions by using different names) that are related through the equation $q = Q(r)$ and its inverse, $r = Q^{-1}(q)$, so that

$$f_1(q)dq = f_2(r)dr \ , \tag{4}$$

then the relationship between the densities is

$$f_2(r) = f_1(q)\left(\frac{dQ}{dr}\right) \ , \tag{5}$$

where $\frac{dQ}{dr} = \frac{dq}{dr}$ is the *Jacobian* of the transformation. Sometimes the Jacobian is taken to be the absolute value of the derivative, $\left|\frac{dQ}{dr}\right|$, as when the relationship between variables is inverse—but the relationship must always be monatomic in the relevant ranges of variables.

Expressing energies in terms of lethargies, the differential inelastic scattering cross section per unit final energy is

$$\frac{\partial \Sigma_{scat}\left(E_o \rightarrow E\right)}{\partial E} = \frac{\Sigma_{scat}(u_o)}{(1-\alpha)E_{ref}} \exp\left(u_o\right) \begin{cases} 1 \text{ for } u_o < u < u_o + \ln\left(\frac{1}{\alpha}\right) \text{ and} \\ 0 \text{ otherwise} \end{cases} \tag{6}$$

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and the differential cross section per unit final lethargy for variable initial lethargy is

$$\frac{\partial \Sigma_{scat}\left(u_o \rightarrow u\right)}{\partial u} = \frac{\partial \Sigma_{scat}\left(E_o \rightarrow E\right)}{\partial E} \left| \frac{dE}{du} \right| =$$

$$= \frac{\Sigma_{scat}(u_o)}{(1-\alpha)} \exp\left(u_o - u\right) \begin{cases} 1 \text{ for } u - \ln\left(\frac{1}{\alpha}\right) < u_o < u \text{ and} \\ 0 \text{ for } u_o < u - \ln\left(\frac{1}{\alpha}\right) \text{ and } u_o > u \end{cases} \quad . \tag{7}$$

Meanwhile,

$$\varphi(\vec{r},E,t) = \frac{1}{E_{ref}} \exp\left(u\right) \varphi(\vec{r},u,t) \text{ and } S(\vec{r},E,t) = \frac{1}{E_{ref}} \exp\left(u\right) S(\vec{r},u,t) \quad , \tag{8}$$

so that, making the substitutions, the diffusion equation becomes

$$\frac{1}{v(u)} \frac{\partial \varphi(\vec{r},u,t)}{\partial t} = D(u) \nabla^2 \varphi(\vec{r},u,t) - \Sigma_{total}(u) \varphi(\vec{r},u,t) +$$

$$+ \frac{1}{(1-\alpha)} \int_{u - \ln\left(\frac{1}{\alpha}\right)}^{u} du' \exp\left(u' - u\right) \Sigma_{scatt}(u') \varphi(\vec{r},u',t) + \quad S(\vec{r},u,t) \ . \tag{9}$$