[Greuling-Goertzel]
(Chapter 1.10.3 in *Elements*)

The Greuling-Goertzel approximation

The description of neutrons slowing down in heavy moderator media is mathematically more difficult than that in hydrogenous media, so requires approximate treatment. First we treat the infinite-medium, steady state case, for which the neutron balance equation, now in terms of the collision density $\psi(u) = \Sigma_{tot}(u)\varphi(u)$, is

$$\psi(u) = \frac{1}{(1-\alpha)} \int_{u-\ln\left(\frac{1}{\alpha}\right)}^{u} \exp\,(u' - u)\frac{\Sigma_{scatt}(u')}{\Sigma_{tot}(u')}\psi(u')du' + S(u) \quad . \tag{1}$$

This is a *Fredholm integral equation of the second kind* because the independent variable u appears in both limits of the integral. We say nothing more about this except that it is not amenable to the simple treatment as for hydrogen (A=1, α = 0). To treat the present case of heavy scatterers (A = > 1, α = > 0), Goertzel and Greuling (1960; Greuling 1952) expanded the flux at lethargy u' in a Taylor series about lethargy u in the integrand, truncated to include only the second-order term, which is the *Greuling-Goertzel approximation,*

$$\psi(u') \approx \psi(u) + \frac{d\psi(u)}{du}(u' - u) + \frac{1}{2}\frac{d^2\psi(u)}{du^2}(u' - u)^2 \quad . \tag{2}$$

The justification is that $\psi(u')$ varies little from $\psi(u)$ when u' is near u. Parks, et al. (1970, 404) discuss this procedure in more detail including higher-order terms. Introducing the truncated expansion produces a second-order linear ordinary differential equation

$$\psi(u) = y_o(u)\psi(u) + y_1(u)\frac{d\psi(u)}{du} + \quad y_2(u)\frac{d^2\psi(u)}{du^2} + S(u) \quad , \tag{3}$$

in which appear the moments of the logarithmic energy loss distribution

$$y_n(u) = \frac{1}{(1-\alpha)}\int_{u-\ln\frac{1}{\alpha}}^{u} \frac{\Sigma_{scatt}(u')}{\Sigma_{tot}(u')}\exp\,(u' - u)(u - u')^n du' \quad . \tag{4}$$

Now, for simplicity, assuming that there is no absorption so that $\frac{\Sigma_{scatt}(u')}{\Sigma_{tot}(u')} = 1$,

$$y_n(u) = \frac{1}{(1-\alpha)}\int_0^{\ln\frac{1}{\alpha}} x^n \exp\,(-x)dx, \tag{5}$$

which are standard integrals, we have, independently of u,

$$y_o = 1\,, \tag{6}$$

$$y_1 = 1 + \frac{\alpha ln(\alpha)}{(1-\alpha)} = \xi\,, \tag{7}$$

and

$$y_2 = \frac{2(1-\alpha) + 2\alpha ln(\alpha) - \alpha ln^2(\alpha)}{(1-\alpha)} = 2\xi\gamma\,, \tag{8}$$

in which

$$\gamma = 1 - \frac{\alpha ln^2(\alpha)}{2\xi(1-\alpha)}\,. \tag{9}$$

In the case $\frac{\Sigma_{scatt}(u')}{\Sigma_{tot}(u')} = 1$, the terms $\psi(u)$ drop out because $y_o = 1$ so that the resulting form of (3) is a first-order ordinary differential equation in $\frac{d\psi(u)}{du}$,

$$y_1 \frac{d\psi}{du} + y_2 \frac{d^2\psi}{du^2} + S(u) = 0\,, \tag{10}$$

in which y_1 and y_2 are constants.

The Greuling-Goertzel equation is an initial-value problem in which the lethargy *u* is a variable analogous to time and the coefficients are independent of *u*, so that the Laplace transform applies. The transformed equation is

$$y_1\big(p\tilde{\psi}(p) - \psi(0)\big) + y_2\big(p^2 - p\psi(0) - \psi'(0)\big) = \;\; -\exp\left(-pu_Q\right)S(u_Q)\,, \tag{11}$$

in which we have taken the source to be a delta-function at lethargy u_Q,

$$S(u) = S(u_Q)\delta(u - u_Q). \tag{12}$$

The Laplace-transformed collision density is $\tilde{\psi}(p) = \mathscr{L}\left[\psi(u)\right]$, and *p* is the *u*-related transform variable. Taking $\psi(0) = 0$ and $\psi'(0) = \left.\frac{d\psi(u)}{du}\right|_0 = 0$, because these bring nothing new to the solution, gives the results, using inverse transforms from Laplace transform-function tables,

$$\tilde{\psi}(p) = -\frac{1}{y_2 p\left(p + \frac{y_1}{y_2}\right)}\exp\left(-pu_Q\right)S(u_Q) \tag{13}$$

, and, for $u > u_O$, (and 0 for $u < u_O$)

$$\psi(u) = \frac{1}{\xi}\left(1 - \exp\left(-\frac{u - u_Q}{2\gamma}\right)\right)S(u_Q)\,. \tag{14}$$

We have not directly addressed the time distribution of neutrons emerging from the surface of a moderator, but experience lends credence to the assertion that it is of the same form as the flux in the infinite medium calculated here. Some useful properties of the time distribution follow from its Laplace transform,

$$\tilde{i}(E, s) = \mathcal{L}[i(E,t)] = \frac{\left(\frac{\xi \Sigma_{scatt} v}{\gamma}\right)^{2/\gamma + 1}}{\left(s + \frac{\xi \Sigma_{scatt} v}{\gamma}\right)^{2/\gamma + 1}} \;, \tag{15}$$

and because $\int_0^\infty i(E,t)dt = \tilde{i}(E, 0) = 1$,

$$\langle t^n \rangle = (-1)^n \frac{d^n \tilde{i}(E, s)}{ds^n}\bigg|_{s=0} \;. \tag{16}$$