[fractal] supplementary to {§14.9.3:480}

Fractals are objects that display the *self-similarity* property; in other words, a fractal appears the same under *any* scale of magnification or miniaturization. This notion is fine for certain mathematical implementations but is counter-intuitive in the material world. However, it has been shown that in real life numerous substances ranging in dimension from microscopic to macroscopic do exhibit self-similar behavior, albeit restricted to sizes within some lower and upper limits, and are considered to be fractals. The fractal systems we encounter in SANS are nanoscale materials, often of low macroscopic density (e.g., porous) and disordered. But a fractal needs not be disordered or random. Of relevance here are two kinds of fractals: mass fractals and surface fractals. The fractal dimension is an important parameter of characterization, measurable by SANS experiments. We use the Euclidean space to describe the presence of fractals and their dimensionality.

Mass fractals

If we try to measure the mass of an object by completely occupying its medium with penetrable spheres of radius r and then count the minimum number of spheres, N, used, the smaller r requires the larger number of spheres. The very property of self-similarity for a mass fractal is expressed as a relation between r and N according to

$$N(r)=\left(\frac{r}{r_0}\right)^{D_m}, \tag{1}$$

where D_{m} is the *mass fractal dimension* and r_0 is the measuring gauge (these spheres with radius r_0 become the primary particles of the fractal). Our goal is to find $g(r)$ from $N(r)=A\int_0^r g(r)4\pi r^2\,dr$ and (1). Differentiation of both equations leads to

$$\begin{aligned} dN(r) &= Ag(r)4\pi r^2 dr \\ &= \frac{D_m}{r_0}\left(\frac{r}{r_0}\right)^{D_m-1} dr, \end{aligned} \tag{2}$$

yielding

$$A=\frac{D_m}{4\pi}\frac{r^{D_m-3}}{r_0^{D_m}}. \tag{3}$$

Next, we calculate the structure factor according to (14.27) and obtain

$$S(Q)=1+\frac{D_m}{r_0^{D_m}}\int_0^\infty r^{D_m-1}\exp\left(-\frac{r}{\xi}\right)\frac{\sin(Qr)}{Qr}dr, \tag{4}$$

where an exponential decay with a cutoff ξ is added to remedy the unphysical behavior of the fractal as $r\to\infty$ or to ensure the condition of $S(Q)\overset{Q\to\infty}{=}1$. The final result is

$$S(Q)+1+\frac{1}{(Qr_0)^{D_m}}\frac{D_m\Gamma(D_m-1)}{\left(1+\frac{1}{Q^2\xi^2}\right)^{(D_m-1)/2}}\sin\left[(D_m-1)\tan^{-1}(Q\xi)\right]. \tag{5}$$

Surface fractals

A surface becomes a fractal when it exhibits the self-similarity property, which can be tested by the same measurement as before[as when?] using spheres of cross-section area πr_0^2. All surface fractals must contain at least one fractal surface. The *surface fractal dimension*, D_s, has meaningful values between 2 and 3. Following a similar argument as before in (1) to (3), we find

$$1-g(r)=\frac{nr^3(r_0/r)^D}{4c(1-c)V}, \tag{6}$$

where c and 1-c are the fractions of dense material and pores, respectively in a sample of volume V, and n is the number of cubes of edge r_0 needed to make a layer covering all points of the fractal surface. The structure factor becomes

$$S(Q)=\pi A'\rho^2\frac{\Gamma(5-D_s)}{Q^{6-D_s}}\sin\left[\frac{\pi(D_s-1)}{2}\right]. \tag{7}$$